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NONLINEAR THEORY AND STABILITY OF THICK SANDWICH SHELLS

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Iu. N. NOVICHKOV
(Moscow)
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A derivation of the fundamental equations of a nonlinear theory of thick sandwich shells is given in tensor form. The shells are fabricated from alternating layers of different stiffness. It is considered that the hypotheses of the refined theory of shells of S. P. Timoshenko are valid for the hard layers, while the soft layers operate under transverse compression and shear. The change in metric during passage from one layer to another is taken into account. Diverse variants of the fundamental equations are presented, including equations for shells of an anisotropic couple-stress continual medium equivalent in an energy sense. The equations are linearized with respect to the membrane state for all the variants. The linearized equations are applied to stability problems of sandwich shells. The problem of the local stability of a cylindrical shell under axial compression is examined as an illustration. The change in the character of the buckling and the magnitude of the critical load is investigated as the relative stiffness of the layers changes.

A study of the mechanical properties of composite materials bonded by high-strength layers, and the investigation of both the integrated and local effects upon deformation of the laminar media can be carried out on the basis of the theory developed in [1]. According to this theory, the behavior of the laminar media is described by a system of differential-difference equations permitting taking account of the discrete properties of the composite laminar medium. A development of this theory is given in [2] in application to sandwich shells. The equations obtained in [2] by using an assumption about the smallness of the displacements and strains can be used for thick shells whose thickness is commensurate with the minimum radius. The thickness of the layers of high stiffness should be small compared to this radius $(h \ll R)$. The assumption of smallness of the displacements should be discarded for the large class of problems associated with finite displacements, and nonlinear equations should be used. In deriving the nonlinear equations the assumption about the commensurability of displacements with the thickness of the hard layers $(u_j \sim h)$ and about the smallness of the displacements as compared with the minimal radius $(u_j \ll R)$ is natural. Moreover, the influence of shears in the hard layers turns out to be essential in some problems.

Nonlinear equations of the theory of thick sandwich shells of regular construction are derived below taking account of transverse shears in the layers of high striffness. As a particular case, the equations presented in [3−5], as well as the nonlinear equations for a single-layered shell [6], can be obtained from these equations. By applying the principle of continualization [7], nonlinear equations are obtained for anisotropic couple-stress media equivalent in an energy sense.

1. Fundamental hypotheses and dependences. Let us consider a shell consisting of alternate layers of high stiffness of thickness h (reinforcing layers) and layers of reduced stiffness of thickness s (matrix layers). For brevity, let us henceforth call them stiff and soft, respectively. Let us consider the layer middle surfaces to be equidistant.

We discard the Kirchhoff-Love hypotheses in constructing the theory for the stiff layers, and apply the hypothesis of the refined theory of shells according to which an element normal to the undeformed middle surface remains rectilinear after deformation, but not perpendicular to the deformed middle surface.

This corresponds to the acceptance of the assumption about the uniform distribution of the transverse shears over the thickness of the stiff layer. Neglecting the transverse shears in the stiff layers results in the Kirchhoff-Love hypotheses for these layers. Moreover, let us consider the displacements of the middle surfaces of the stiff layers $v_j^{(k)}$ to be negligibly small as compared to the minimum radius of curvature of the layer and the scale of variation of the state of the middle surface, although still commensurate with the thickness of the stiff layer h. We assume the hypothesis of a linear change in displacement over the thickness for the soft layers.

Let us refer the laminar shell to an orthogonal x^1, x^2, x^3 coordinate system such that the middle surfaces of the stiff layers are the coordinates $x^3 = \mathrm{const}$. The surfaces $x^\alpha = \mathrm{const}$ $(\alpha = 1,2)$ are orthogonal to these surfaces. Moreover, let us introduce local coordinate systems for each layer as a consequence of the parallel transfer of the mentioned system along x^3.

We obtain an expression for the displacements by starting from the assumption about a uniform distribution of the transverse shears along the thickness of the stiff layer. As a result of integration and the linearization admissible in connection with the hypotheses assumed, we obtain the following expressions for the displacement components in the stiff layers:

$$u_\alpha^{(k)} = (\delta_\alpha^\beta + z^{(k)} b^{(k)\beta}_{\ \ \alpha})(v_\beta^{(k)} - z^{(k)}\xi_\beta^{(k)}) \tag{1.1}$$
$$w^{(k)} = v_3^{(k)}, \qquad x^{(k)3} = z^{(k)}$$

Here $v_\alpha^{(k)}$, $v_3^{(k)}$ are covariant components of the middle surface displacement vector of the kth stiff layer, $b^{(k)\alpha}_{\ \ \beta}$ are mixed components of the curvature tensor, $\xi_\beta^{(k)}$ are components of the surface vector of the slopes of the normals. In the case of the validity of the Kirchhoff-Love hypotheses

$$\xi_\beta^{(k)} = \varphi_\beta^{(k)} = \nabla_\beta^{(k)} w^{(k)} - b^{(k)\gamma}_{\ \ \beta} v_\gamma^{(k)} \tag{1.2}$$

should be introduced in (1.1) in place of $\xi_\beta^{(k)}$ for the strain of the stiff layers (the assumption is valid for the higher degree of flexibility of the soft layers), where $\nabla_\beta^{(k)}$ is the symbol of covariant differentiation on the middle surface of the kth stiff layer.

We find the strain components of the stiff layers by means of the formula

$$\varepsilon_{\alpha\beta}^{(k)} = {}^1/_2 (\overset{3}{\nabla}{}_\alpha^{(k)} u_\beta^{(k)} + \overset{3}{\nabla}{}_\beta^{(k)} u_\alpha^{(k)} + \overset{3}{\nabla}{}_\alpha^{(k)} u^{(k)l}\, \overset{3}{\nabla}{}_\beta^{(k)} u_l^{(k)}) \tag{1.3}$$

As usual, the Greek subscripts take on the values 1, 2 and the Latin subscripts 1, 2, 3. Here $\overset{3}{\nabla}{}_\alpha^{(k)}$ in (1.3) is the symbol of covariant differentiation in the space surrounding the middle surface of the kth stiff layer. Using (1.1) and the rules of tensor calculus, we find that the strain components are

$$\varepsilon_{\alpha\beta}^{(k)} = e_{\alpha\beta}^{(k)} - z^{(k)} \varkappa_{\alpha\beta}^{(k)} \tag{1.4}$$

to the accuracy of linear terms in the normal coordinate $z^{(k)}$, where the strain tensor components of the middle surface $e_{\alpha\beta}^{(k)}$ and the change in the curvature $\varkappa_{\alpha\beta}^{(k)}$ are determined by the expressions

$$e_{\alpha\beta}^{(k)} = \overset{\circ}{e}{}_{\alpha\beta}^{(k)} + {}^1/{}_2\psi^{(k)\gamma}{}_{\alpha}\psi_{\beta\gamma}^{(k)} + {}^1/{}_2\varphi_\alpha^{(k)}\varphi_\beta^{(k)}$$
$$\varkappa_{\alpha\beta}^{(k)} = {}^1/{}_2\,(\nabla_\alpha^{(k)}\xi_\beta^{(k)} + \nabla_\beta^{(k)}\xi_\alpha^{(k)} - b^{(k)\gamma}{}_{\alpha}\psi_{\beta\gamma}^{(k)} - b^{(k)\gamma}{}_{\beta}\psi_{\alpha\gamma}^{(k)}) \tag{1.5}$$

Here $\varphi_\alpha{}^{(k)}$ is determined in conformity with (1.2), the quantities $\psi_{\alpha\beta}^{(k)}$ are the sum of components of the antisymmetric tensor of the rotation around the normal $\theta_{\alpha\beta}^{(k)}$ and the surface tensor $\overset{\circ}{e}{}_{\alpha\beta}^{(k)}$ describing the linear part of the tangential strains of the middle surface

$$\psi_{\alpha\beta}^{(k)} = \theta_{\alpha\beta}^{(k)} + \overset{\circ}{e}{}_{\alpha\beta}^{(k)}$$
$$\theta_{\alpha\beta}^{(k)} = {}^1/{}_2\,(\nabla_\alpha^{(k)}v_\beta^{(k)} - \nabla_\beta^{(k)}v_\alpha^{(k)}) \tag{1.6}$$
$$\overset{\circ}{e}{}_{\alpha\beta}^{(k)} = {}^1/{}_2\,(\nabla_\alpha^{(k)}v_\beta^{(k)} + \nabla_\beta^{(k)}v_\alpha^{(k)}) + b_{\alpha\beta}^{(k)}w^{(k)}$$

Nonlinear terms of the type

$$\psi^{(k)}{}_{\beta}{}^{\gamma}\,(\nabla_\gamma^{(k)}\varphi_\alpha^{(k)} - 2b^{(k)}{}_{\gamma}^{\mu}\psi_{\alpha\mu}^{(k)})$$

which are higher order infinitesimals are omitted in the expression for $\varkappa_{\alpha\beta}^{(k)}$. Retention of these terms would result in the need to retain square terms in $z^{(k)}$ in (1.4) and some nonlinear terms in (1.1), which would not correspond to the assumptions accepted herein. Such a constraint on the nonlinear terms is customary for the nonlinear theory of single-layer shells [6]. The transverse shear strain components in the stiff layers are the following within the scope of the hypotheses assumed:

$$\varepsilon_{\alpha3}^{(k)} = {}^1/{}_2\,(\varphi_\alpha^{(k)} - \xi_\alpha^{(k)}) \tag{1.7}$$

By passing to the examination of soft layers in conformity with the hypothesis about a linear change in displacements along the thickness of the soft layer, we obtain

$$u_\alpha^{[k]} = \frac{1}{2}\left[u_\alpha^{(k)}\left(\frac{h}{2}\right) + u_\alpha^{(k+1)}\left(-\frac{h}{2}\right)\right] + \frac{z^{[k]}}{s}\left[u_\alpha^{(k+1)}\left(-\frac{h}{2}\right) - u_\alpha^{(k)}\left(\frac{h}{2}\right)\right]$$
$$w^{[k]} = \frac{1}{2}(w^{(k)} + w^{(k+1)}) + \frac{z^{[k]}}{s}\,(w^{(k+1)} - w^{(k)}) \tag{1.8}$$

Inserting (1.1) into (1.8) by means of formulas analogous to (1.3), we evaluate the essential strain tensor components of the soft layers. By estimating the order of the terms in the expression obtained and by omitting higher order terms, we arrive at relationships for the average shear and transverse strain components in the soft layers

$$\varepsilon_{\alpha3}^{[k]} = \frac{1}{2s}\left[\frac{s}{2}\nabla_\alpha^{[k]}(w^{(k+1)} + w^{(k)}) + (v_\alpha^{(k+1)} - v_\alpha^{(k)}) - \right.$$
$$\left.\frac{h+2s}{2}\,b^{[k]\beta}{}_{\alpha}\,(v_\beta^{(k+1)} + v_\beta^{(k)}) + \frac{h}{2}\,(\xi_\alpha^{(k+1)} + \xi_\alpha^{(k)})\right] \tag{1.9}$$
$$\varepsilon_{33}^{[k]} = \frac{1}{s}(w^{(k+1)} - w^{(k)})$$

If the Kirchhoff-Love hypotheses are valid for the stiff layers, then expressions agreeing with an analogous expression in [2] are obtained instead of (1.9)

$$\varepsilon_{\alpha3}^{[k]} = {}^1/{}_2 s^{-1}\,[c\nabla_\alpha^{[k]}(w^{(k+1)} + w^{(k)}) + (v_\alpha^{(k+1)} - v_\alpha^{(k)}) - \tag{1.10}$$
$$2b^{[k]\beta}{}_{\alpha}c\,(v_\beta^{(k+1)} + v_\beta^{(k)})] \qquad (2c = h + s)$$

The stresses in the stiff and soft layers are computed in conformity with Hooke's law. Formulas for the plane state of stress

$$\sigma^{(k)\alpha\beta} = \lambda^{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta}^{(k)}, \qquad \sigma^{(k)\alpha 3} = 2Ga^{(k)\alpha\beta}\varepsilon_{\beta 3}^{(k)} \tag{1.11}$$
$$\sigma^{[k]\alpha 3} = 2G_m a^{[k]\alpha\beta}\varepsilon_{\beta 3}^{[k]}, \qquad \sigma^{[k]33} = E_m \varepsilon_{33}^{[k]}$$

hence hold for the soft layers. Here $\lambda^{\alpha\beta\gamma\delta}$ are the covariant components of the elastic constants tensor of the plane state of stress of a stiff layer, G is the shear modulus of the stiff layer material, E_m and G_m are the transversal modulus and shear modulus of the soft layers, $a^{(k)\alpha\beta}$ and $a^{[k]\alpha\beta}$ are contravariant components of the middle-surface metric tensor of the kth stiff and soft layers. Let us note that (1.11) includes the cases of both isotropic and anisotropic stiff layers.

2. Equations of nonlinear sandwich shell theory. Let us use the Lagrange variational principle to derive the fundamental equations. It is hence necessary to obtain an expression for the strain potential energy of a laminar shell and an expression for the potential of the applied forces. Introducing the energy stress resultants and moments associated with the stresses referred to the middle surfaces by the formulas

$$\mathbf{N}^{(k)\alpha\beta} = h\sigma_0^{(k)\alpha\beta}, \qquad \mathbf{M}^{(k)\alpha\beta} = h^3\sigma_{00}^{(k)\alpha\beta}/12 \tag{2.1}$$
$$\mathbf{S}^{(k)\alpha} = h\sigma^{(k)\alpha 3}, \qquad \sigma^{(k)\alpha\beta} = \sigma_0^{(k)\alpha\beta} - z^{(k)}\sigma_{00}^{(k)\alpha\beta}$$

let us write the expression for the potential strain energy

$$U' = \sum_{k=1}^{n} U_{(k)} = \frac{1}{2}\sum_{k=1}^{n}\iint_{\Omega_{(k)}} (\mathbf{N}^{(k)\alpha\beta}e_{\alpha\beta}^{(k)} + \mathbf{M}^{(k)\alpha\beta}\varkappa_{\alpha\beta}^{(k)} + \mathbf{S}^{(k)\alpha}\varepsilon_{\alpha 3}^{(k)})\, d\Omega_{(k)} \tag{2.2}$$

The potential strain energy of the soft layers is defoned as follows:

$$U'' = \sum_{k=1}^{n-1} U_{[k]} = \frac{1}{2}\sum_{k=1}^{n-1}\iint_{\Omega_{[k]}} (-2Q^{[k]\alpha}\varepsilon_{\alpha 3}^{[k]} + N^{[k]}\varepsilon_{33}^{[k]})\, d\Omega_{[k]} \tag{2.3}$$

$$Q^{[k]\alpha} = -{}^1/_2 s\sigma^{[k]\alpha 3}, \qquad N^{[k]} = s\sigma^{[k]33}$$

Here $Q^{[k]\alpha}$, $N^{[k]}$ have the meaning of transverse forces due to taking account of the shears, and tensile forces associated with the transverse strain of a soft layer. In calculating the potential of the applied forces, let us consider, without limiting the generality, that the applied forces and moments act on the stiff layers

$$\Pi = -\sum_{k=1}^{n}\Big[\iint_{\Omega_{(k)}} (q_{(k)}^3 w^{(k)} + q_{(k)}^\alpha v_\alpha^{(k)})\, d\Omega_{(k)} + \tag{2.4}$$
$$\oint_{\Gamma_{(k)}} (-Q_{(k)}w^{(k)} + M_{(k)}^n\xi_n^{(k)} + M_{(k)}^l\xi_l^{(k)} + N_{(k)}^\alpha v_\alpha^{(k)})\, d\Gamma_{(k)}\Big] + \text{const}$$

Let us introduce the customary stress resultants and moments, in addition to the energetically derived stress resultants and moments introduced above. Their interrelation is given by $N^{(k)\alpha\beta} = \mathbf{N}^{(k)\alpha\beta} + {}^1/_2 b_{\ \nu}^{(k)\alpha}\mathbf{M}^{(k)\beta\nu} - {}^1/_2\, b_{\ \nu}^{(k)\beta}\mathbf{M}^{(k)\alpha\nu} + \mathbf{N}^{(k)\alpha\gamma}\psi_\gamma^{(k)\beta}$

$$M^{(k)\alpha\beta} = \mathbf{M}^{(k)\alpha\beta}, \qquad S^{(k)\alpha} = \mathbf{S}^{(k)\alpha} \tag{2.5}$$

Let us note that if a simpler expression is taken in place of (1.5) for the curvature $(\psi^{(k)}_{\alpha\beta} \approx \theta^{(k)}_{\alpha\beta})$, in conformity with the approximation in [6] for a single-layer shell, then we must take

$$N^{(k)\alpha\beta} = \mathbf{N}^{(k)\alpha\beta} + {}^1/_2 b^{(k)\beta}_{\ \gamma} M^{(k)\alpha\gamma} - {}^1/_2 b^{(k)\alpha}_{\ \gamma} M^{(k)\beta\gamma} + \mathbf{N}^{(k)\alpha\gamma}\theta^{(k)\beta}_{\ \gamma} \tag{2.6}$$

instead of the first equality in (2.5). The energy and customary stress resultants agree in case the hypotheses corresponding to shallow shells are valid.

Application of the Lagrange principle with (2.2) – (2.4) and the notation (2.5) used, results in the fundamental finite-difference-differential equations for sandwich shells

$$\begin{gathered}\nabla^{(k)}_\beta N^{(k)\alpha\beta} + b^{(k)\alpha}_{\ \gamma}\mathbf{N}^{(k)\beta\gamma}\varphi^{(k)}_\beta + b^{(k)\alpha}_{\ \beta}S^{(k)\beta} + s^{-1}(t_k^* Q^{[k]*\alpha} - t_k^{**}Q^{[k-1]**\alpha}) - \\ \frac{h+2s}{2s} b^{[k]\alpha}_{\ \beta}(t_k^* Q^{[k]*\beta} + t_k^{**}Q^{[k-1]**\beta}) + q^\alpha_{(k)} = 0 \qquad (2.7) \\ b^{(k)}_{\alpha\beta}N^{(k)\alpha\beta} - \nabla^{(k)}_\alpha(\mathbf{N}^{(k)\alpha\beta}\varphi^{(k)}_\beta) - \nabla^{(k)}_\alpha S^{(k)\alpha} + \nabla^{(k)}_\alpha(t_k^* Q^{[k]*\alpha} + t_k^{**}Q^{[k-1]**\alpha}) - \\ s^{-1}(t_k^* N^{[k]} - t_k^{**}N^{[k-1]}) - q^3_{(k)} = 0 \\ \nabla^{(k)}_\beta M^{(k)\alpha\beta} + S^{(k)\alpha} + {}^1/_2 hs^{-1}(t_k^* Q^{[k]*\alpha} - t_k^{**}Q^{[k-1]**\alpha}) = 0\end{gathered}$$

Here

$$\begin{gathered}t_k{}^* = \frac{d\Omega_{[k]}}{d\Omega_{(k)}} = \left(\frac{a^{[k]}}{a^{(k)}}\right)^{1/2}, \quad t_k{}^{**} = \left(\frac{a^{[k-1]}}{a^{(k)}}\right)^{1/2}, \quad t_n{}^* = t_1{}^{**} = 0 \\ Q^{[k]*\alpha} = (\delta^\alpha_\beta - cb^{[k]\alpha}_{\ \ \beta})\, Q^{[k]\beta}, \qquad Q^{[k-1]**\alpha} = (\delta^\alpha_\beta + cb^{[k-1]\alpha}_{\ \ \beta})\, Q^{[k-1]\beta}\end{gathered} \tag{2.8}$$

The last relations result from the general rules for transformation of surface tensors. The natural boundary conditions are also furnished by the variational principle. For example, for the edge $x^1 = \text{const}$ they are

$$\begin{gathered}\sqrt{a^{(k)}}\, N^{(k)\alpha 1} = \sqrt{a^{(k)}_{22}}\, N^\alpha_{(k)}, \qquad \sqrt{a^{(k)}}\, M^{(k)\alpha 1} = \sqrt{a^{(k)}_{22}}\, M^{\alpha 1}_{(k)} \\ \sqrt{a^{(k)}}\,(t_k^* Q^{[k]*1} + t_k^{**}Q^{[k-1]**1} + \mathbf{N}^{(k)\beta 1}\varphi^{(k)}_\beta + S^{(k)1}) = \sqrt{a^{(k)}_{22}}\, Q^1_{(k)}\end{gathered} \tag{2.9}$$

If the transverse shears in the stiff layers are negligible, then the components $\xi_\alpha{}^{(k)}$ must be replaced in all the expressions starting with (1.1) and up to (2.4) in conformity with (1.2) and we must set $S^{(k)} = 0$ in the expression for the potential energy U' of the stiff layers. Application of the Lagrange principle results in the following equations:

$$\begin{gathered}\nabla^{(k)}_\beta N^{(k)\alpha\beta} - b^{(k)\alpha}_{\ \beta}\nabla^{(k)}_\gamma M^{(k)\beta\gamma} + s^{-1}(t_k^* Q^{[k]*\alpha} - t_k^{**}Q^{[k-1]**\alpha}) - \\ 2cs^{-1}b^{[k]\alpha}_{\ \beta}(t_k^* Q^{[k]*\beta} + t_k^{**}Q^{[k-1]**\beta}) + \mathbf{N}^{(k)\gamma\beta}b^{(k)\alpha}_{\ \gamma}\varphi^{(k)}_\beta + q^\alpha_{(k)} = 0 \qquad (2.10) \\ b^{(k)}_{\alpha\beta}N^{(k)\alpha\beta} + \nabla^{(k)}_\alpha\nabla^{(k)}_\beta M^{(k)\alpha\beta} + \nabla^{(k)}_\alpha[cs^{-1}(t_k^* Q^{[k]*\alpha} + t_k^{**}Q^{[k-1]**\alpha}) - \\ s^{-1}(t_k^* N^{[k]} - t_k^{**}N^{[k-1]}) - \nabla^{(k)}_\alpha(\mathbf{N}^{(k)\alpha\beta}\varphi^{(k)}_\beta) - q^3_{(k)} = 0\end{gathered}$$

and the natural boundary conditions $(x^1 = \text{const})$

$$\begin{gathered}\sqrt{a^{(k)}}\,(N^{(k)\alpha 1} - b^{(k)\alpha}_{\ 2}M^{(k)21}) = \sqrt{a^{(k)}_{22}}\,(N^\alpha_{(k)} - b^{(k)\alpha}_{\ 2}M^{21}_{(k)}), \\ \sqrt{a^{(k)}}\, M^{(k)11} = \sqrt{a^{(k)}_{22}}\, M^{11}_{(k)}\end{gathered}$$

$$\sqrt{a^{(k)}}\,[cs^{-1}(t_k^{*}Q^{[k]*1}+t_k^{**}Q^{[k-1]**1})+\nabla_\beta^{(k)}M^{(k)1\beta}-N^{(k)\beta 1}\varphi_\beta^{(k)}+ \quad (2.11)$$
$$\frac{1}{\sqrt{a^{(k)}}}\frac{\partial}{\partial x^2}(\sqrt{a^{(k)}}M^{(k)12})]=\sqrt{a_{22}^{(k)}}\left[Q_{(k)}^1+\frac{1}{\sqrt{a_{22}^{(k)}}}\frac{\partial}{\partial x^2}(\sqrt{a_{22}^{(k)}}M_{(k)}^{12})\right]$$

The quantities with the asterisks in these equations and boundary conditions are defined according to (2.8), as before.

Let us consider a further simplification of (2.7), (2.10). If the thickness of the whole shell H is small compared to the characteristic radius, then the shell can be considered thin, and the change in the metric can be neglected. In this case, we set

$$a_{\alpha\beta}^{(k)}=a_{\alpha\beta},\quad b_{\alpha\beta}^{(k)}=b_{\alpha\beta},\quad Q^{[k]\alpha}=Q^{[k]*\alpha}=Q^{[k]**\alpha}$$
$$\nabla_\alpha^{(k)}=\nabla_\alpha,\quad t_k^*=\eta_{kn},\quad t_k^{**}=\eta_{k1},\quad \eta_{kj}=1-\delta_{kj}$$

in (2.7), (2.10) and in the appropriate boundary conditions. Introduction of hypotheses corresponding to shallow shells gives an essential simplification in the equations. In this case the nonlinear terms containing $\psi_{\alpha\beta}$ in (1.5) should be neglected, and φ_α should be replaced by $\nabla_\alpha w$ in the remaining nonlinear terms. Moreover, terms containing b_α^β should be neglected in the expressions for the tensor components of the change in curvature. As an illustration, let us present the form of (2.10) for shallow shells

$$\begin{gathered}\nabla_\beta N^{(k)\alpha\beta}+s^{-1}(\eta_{kn}Q^{[k]\alpha}-\eta_{k1}Q^{[k-1]\alpha})+q_{(k)}^\alpha=0\\ b_{\alpha\beta}N^{(k)\alpha\beta}+\nabla_\alpha\nabla_\beta M^{(k)\alpha\beta}+cs^{-1}(\eta_{kn}Q^{[k]\alpha}+\eta_{k1}Q^{[k+1]\alpha})-\\ s^{-1}(\eta_{kn}N^{[k]}-\eta_{k1}N^{[k-1]})-\nabla_\alpha(N^{(k)\alpha\beta}\nabla_\beta w)-q_{(k)}^3=0\end{gathered} \quad (2.12)$$

In this case the stress resultants and moments agree with the energetically derived stresses and moments and are defined according to (2.1). Neglecting the tangential inertia forces is natural in analyzing dynamics problems for the case presented. Further simplifications are obtained when neglecting deformations of the normals in the soft layers and introducing the total stress resultant functions. The equations of [4] can hence be obtained from (2.7).

3. Passage to a continuum couple-stress anisotropic medium. When the number of layers is sufficiently high, replacement of the discrete laminar medium by a continuum, anisotropic, couple-stress medium is possible. Following [7], let us perform the operation of continualization for the cases examined above.

Let us first examine the case when the Kirchhoff-Love hypotheses are valid for the stiff layers. We have for the strain tensor components $(\alpha,\ \beta=1,2)$

$$e_{\alpha\beta}={}^1/_2(\nabla_\alpha v_\beta+\nabla_\beta v_\alpha+\nabla_\alpha v^\gamma\nabla_\beta v_\gamma+\nabla_\alpha w\nabla_\beta w) \quad (3.1)$$

Here and throughout Sect. 3 ∇_α denotes covariant differentiation in the space surrounding the coordinate surface $x^3=0$. The metric tensor components for this space are $g_{33}=1$, $g_{\alpha 3}=0$, so that the three-index Christoffel symbols containing the index 3 twice and more equal zero: $G_{33}^k=G_{3\alpha}^3=G_{\alpha 3}^3=0$.

On the basis of (1.2) and (1.5), the tensor components characterizing the curvature of an element of a couple-stress medium are:

$$\varkappa_{\alpha\beta}=\nabla_\alpha\nabla_\beta w-{}^1/_2\nabla_\alpha(G_{\beta 3}^\gamma v_\gamma)-{}^1/_2\nabla_\beta(G_{\alpha 3}^\gamma v_\gamma) \quad (3.2)$$

The transverse shears and strains are defined as follows:

$$\varepsilon_{\alpha 3} = \frac{1}{2(1-\psi)}(\nabla_\alpha w + \nabla_3 v_\alpha - G^\beta_{\alpha 3} v_\beta), \quad \varepsilon_{33} = \frac{1}{1-\psi}\nabla_3 w \tag{3.3}$$

where $\psi = h / 2c$ has the meaning of a reinforcing coefficient. The stress and moment components are determined by the formulas

$$\sigma^{\alpha\beta} = \psi\lambda^{\alpha\beta\gamma\delta} e_{\gamma\delta}, \qquad \mu^{\alpha\beta} = \psi\frac{h^2}{12}\lambda^{\alpha\beta\gamma\delta}\varkappa_{\gamma\delta} \tag{3.4}$$

$$\sigma^{33} = E_m\varepsilon_{33}, \qquad \sigma^{\alpha 3} = 2G_m(1-\psi)g^{\alpha\beta}\varepsilon_{\beta 3}$$

Performing the fundamental operation of energy continualization, the replacement of the sum in the potential energy expression by an integral, we obtain

$$U = \frac{1}{2}\iiint\limits_V (\sigma^{\alpha\beta}e_{\alpha\beta} + \mu^{\alpha\beta}\varkappa_{\alpha\beta} + 2\sigma^{\alpha 3}\varepsilon_{\alpha 3} + \sigma^{33}\varepsilon_{33})\, dV$$

Application of the Lagrange principle results in the equations (q^α and q^3 are mass force components)

$$\nabla_\beta\sigma^{\alpha\beta} - \nabla_\beta(G^\alpha_{\gamma 3}\mu^{\beta\gamma}) + \nabla_\beta(\sigma^{\gamma\beta}\nabla_\gamma v^\alpha) - \frac{1}{1-\psi}G^\alpha_{\beta 3}\sigma^{\beta 3} - \frac{1}{1-\psi}\nabla_3\sigma^{\alpha 3} + q^\alpha = 0$$

$$\nabla_\alpha\nabla_\beta\mu^{\alpha\beta} - \frac{1}{1-\psi}(\nabla_\alpha\sigma^{\alpha 3} + \nabla_3\sigma^{33}) - \nabla_\beta(\sigma^{\alpha\beta}\nabla_\alpha w) - q^3 = 0 \tag{3.5}$$

The equations obtained differ from the ordinary equations of couple-stress elasticity theory [8] because of the structural anisotropy of the medium relative to the couple stresses.

Normal stresses $\sigma_{\alpha\beta}{}^\circ$ and shear stresses $\sigma_{3\alpha}{}^\circ$, as well as the moments $m^{\alpha\beta}$ can be given on the faces x^α = const. Only the normal $\sigma_{33}{}^\circ$ and shear stresses $\sigma_{33}{}^\circ$ ($\sigma_{\alpha 3}{}^\circ \neq \sigma_{3\alpha}{}^\circ$) can be assigned on the face x^3 = const.

The natural conditions for the edge x^1 = const are

$$\sqrt{g}\,[\sigma^{\alpha 1} - \mu^{21}G_{23}{}^\alpha + \sigma^{\gamma 1}\nabla_\gamma v^\alpha] = (\sigma_{11}{}^\circ - G_{23}{}^\alpha m^{21})\sqrt{g_{22}}$$

$$\sqrt{g}\,\mu^{11} = \sqrt{g_{22}}\,m^{11}, \qquad \sqrt{g}\left[\frac{1}{1-\psi}\sigma^{31} - \nabla_\beta\mu^{\beta 1} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^2}(\sqrt{g}\,\mu^{21}) + \right. \tag{3.6}$$

$$\left.\sigma^{\alpha 1}\nabla_\alpha w\right] = \sqrt{g_{22}}\left[\sigma_{31}{}^\circ + \frac{1}{\sqrt{g_{22}}}\frac{\partial}{\partial x^2}(\sqrt{g_{22}}\,m^{21})\right]$$

For the edge x^3 = const we have

$$\frac{1}{1-\psi}\sigma^{\alpha 3} = \sigma_{\alpha 3}{}^\circ, \qquad \frac{1}{1-\psi}\sigma^{33} = \sigma_{33}{}^\circ \tag{3.7}$$

Now, let us turn to the case of rejection of the Kirchhoff-Love hypotheses for the stiff layers of a multilayered medium. Upon passing to the continuous medium it is necessary to introduce an additional field of local rotations characterized by the components ξ_α. It is necessary to take

$$\varkappa_{\alpha\beta} = {}^1/_2\,[\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha - \nabla_\alpha(G^\gamma_{\beta 3}v_\gamma) - \nabla_\beta(G^\gamma_{\alpha 3}v_\gamma)] \tag{3.8}$$

instead of (3.2) for the curvature tensor components of the elements of the medium.

The expression for the potential energy is converted into

$$U = \frac{1}{2}\iiint\limits_V [\sigma^{\alpha\beta}e_{\alpha\beta} + \mu^{\alpha\beta}\varkappa_{\alpha\beta} + 2\sigma^{\alpha 3}\varepsilon_{\alpha 3} + \sigma^{33}\varepsilon_{33} + S^\alpha(\nabla_\alpha w - \xi_\alpha)]\, dV \tag{3.9}$$

$$S^\alpha = \psi G(\nabla_\alpha w - \xi_\alpha)$$

Here S^α are the additional shears in the equivalent medium due to the shears in the stiff layers of a discrete laminar medium. We find from the Lagrange principle

$$\nabla_\beta \sigma^{\alpha 3} - \nabla_\beta (G^\alpha_{\gamma 3}\mu^{\beta\gamma}) - \nabla_\beta (\sigma^{\gamma\beta}\nabla_\gamma v^\alpha) - \frac{1}{1-\psi}\nabla_3\sigma^{\alpha 3} - \frac{1}{1-\psi} G^\alpha_{\beta 3}\sigma^{\beta 3} + q^\alpha = 0$$
$$\frac{1}{1-\psi}\nabla_\alpha\sigma^{\alpha 3} + \frac{1}{1-\psi}\nabla_3\sigma^{33} + \nabla_\alpha S^\alpha + \nabla_\beta(\sigma^{\alpha\beta}\nabla_\alpha w) + q^3 = 0 \qquad (3.10)$$
$$\nabla_\beta \mu^{\alpha\beta} + S^\alpha = 0$$

The natural boundary conditions for $x^1 = \text{const}$ become

$$\sqrt{g}\,[\sigma^{\alpha 1} - G_{23}{}^\alpha\mu^{21} + \sigma^{\gamma 1}\nabla_\gamma v^\alpha] = \sqrt{g_{22}}\,(\sigma_{11}{}^\circ - G_{23}{}^\alpha m^{21}) \qquad (3.11)$$
$$\sqrt{g}\,\mu^{\alpha 1} = \sqrt{g_{22}}\, m^{\alpha 1}, \qquad \sqrt{g}\left(\frac{1}{1-\psi}\sigma^{31} + \sigma^{\alpha 1}\nabla_\alpha w + S^1\right) = \sqrt{g_{22}}\,\sigma_{31}{}^\circ$$

The conditions for the edge $x^3 = \text{const}$ retain the form (3.7).

4. Linearization of the equations. The simplest case of linearization of the equations obtained will hold if the initial state of the shell is undeformed. To linearize (2.7), (2.10), (3.5) and (3.10) and the appropriate boundary conditions, it is sufficient to omit the nonlinear terms. Hence, (2.10) go over into the equations in [2].

Linearization with respect to the membrane state in the shell is of interest. The equations obtained are hence the initial equations in an investigation of the stability of sandwich shells. Let us show how the linearization is carried out in this case by using (2.10) as an example, and let us then present the linearized equations for other cases.

The equations

$$\nabla^{(k)}_\beta N^{(k)\alpha\beta} + \mathbf{N}^{(k)\gamma\beta} b^{(k)\alpha}_{\gamma}\varphi^{(k)}_\beta + q^\alpha_{(k)} = 0 \qquad (4.1)$$
$$b^{(k)}_{\alpha\beta} N^{(k)\alpha\beta} - s^{-1}(t_k{}^* N^{[k]} - t_k{}^{**} N^{[k-1]}) - \nabla^{(k)}_\alpha (\mathbf{N}^{(k)\alpha\beta}\varphi^{(k)}_\beta) - q^3_{(k)} = 0$$

correspond to the membrane state in a sandwich shell. Let the solution of this system be

$$\mathbf{N}^{(k)\alpha\beta} = F^{(k)\alpha\beta}, \quad N^{[k]} = N_0^{[k]}, \quad \varphi^{(k)}_\beta = \varphi^{(k)}_{0\beta}$$
$$\psi^{(k)}_{\alpha\beta} = \psi^{(k)}_{0\alpha\beta}, \quad v^{(k)}_\alpha = v^{(k)}_{0\alpha}, \quad w^{(k)} = w^{(k)}_0 \qquad (4.2)$$

Let us seek the solution of (2.10) as the sum of two members, one of which is (4.2) and the other corresponds to a deviation from the initial membrane state

$$\mathbf{N}^{(k)\alpha\beta} = F^{(k)\alpha\beta} + \mathbf{N}^{(k)\alpha\beta}_*, \qquad N^{[k]} = N_0^{[k]} + N_*^{[k]}$$
$$\varphi^{(k)}_\beta = \varphi^{(k)}_{0\beta} + \varphi^{(k)}_{*\beta}, \qquad \psi^{(k)}_{\alpha\beta} = \psi^{(k)}_{0\alpha\beta} + \psi^{(k)}_{*\alpha\beta} \qquad (4.3)$$
$$v^{(k)}_\alpha = v^{(k)}_{0\alpha} + v^{(k)}_{*\alpha}, \qquad w^{(k)} = w^{(k)}_0 + w^{(k)}_*$$
$$Q^{[k]\alpha} = Q^{[k]\alpha}_*, \qquad M^{(k)\alpha\beta} = M^{(k)\alpha\beta}_*$$

Let us substitute (4.3) into (2.10) and let us linearize relative to the deviations taking into account that (4.2) is a solution of (4.1). Upon linearizing, we shall consider, as is customary in the theory of shell stability, that terms of the type $\mathbf{N}^{(k)\alpha\beta}_* \varphi^{(k)}_{0\beta}$ can be neglected in comparison with the terms $F^{(k)\alpha\beta}\varphi^{(k)}_{*\beta}$. In place of the stress resultants $N^{(k)\alpha\beta}$ the following should be inserted into (2.10)

$$N^{(k)\alpha\beta} = N_0^{(k)\alpha\beta} + N_*^{(k)\alpha\beta} + F^{(k)\alpha\gamma}\psi^{(k)\beta}_{*\ \gamma} \qquad (4.4)$$

After linearizing, we arrive at the system of equations (we omit the symbol * of the deviation from the membrane state)

$$\nabla_\beta^{(k)} N^{(k)\alpha\beta} - b^{(k)\alpha}{}_\beta \nabla_\gamma^{(k)} M^{(k)\beta\gamma} + \frac{1}{s}(t_k^* Q^{[k]*\alpha} - t_k^{**} Q^{[k-1]**\alpha}) -$$
$$\frac{2c}{s} b^{[k]\alpha}{}_\beta (t_k^* Q^{[k]*\beta} + t_k^{**} Q^{[k-1]**\beta}) + \nabla_\beta (F^{(k)\alpha\gamma} \psi^{(k)\beta}_\gamma) + F^{(k)\gamma\beta} b^{(k)\alpha}{}_\gamma \varphi_\beta^{(k)} + q_{(k)}^\alpha = 0$$
$$b_{\alpha\beta}^{(k)} N^{(k)\alpha\beta} + \nabla_\alpha^{(k)} \nabla_\beta^{(k)} M^{(k)\alpha\beta} + \nabla_\alpha^{(k)} \left[\frac{c}{s}(t_k^* Q^{[k]*\alpha} + t_k^{**} Q^{[k-1]**\alpha})\right] - \quad (4.5)$$
$$\frac{1}{s}(t_k^* N^{[k]} - t_k^{**} N^{[k-1]}) - \nabla_\alpha^{(k)} (F^{(k)\alpha\beta} \varphi_\beta^{(k)}) + b_{\alpha\beta}^{(k)} F^{(k)\alpha\gamma} \psi_\gamma^{(k)\gamma} - q_{(k)}^3 = 0$$

Here $q_{(k)}^\alpha$ and $q_{(k)}^3$ are understood to be additional stress resultants. The quantities in (4.5) are calculated in conformity with the linear relations (1.5). The boundary conditions are also transformed in an analogous manner. The linearized modification for (2.7) relative to the membrane state is

$$\nabla_\beta^{(k)} N^{(k)\alpha\beta} + b^{(k)\alpha}{}_\gamma F^{(k)\beta\gamma} \varphi_\beta^{(k)} + b^{(k)\alpha}{}_\beta S^{(k)\beta} + \nabla_\beta^{(k)} (F^{(k)\alpha\gamma} \psi_\gamma^{(k)\beta}) + \quad (4.6)$$
$$\frac{1}{s}(t_k^* Q^{[k]*\alpha} - t_k^{**} Q^{[k-1]**\alpha}) - \frac{h+2s}{2s} b^{[k]\alpha}{}_\beta (t_k^* Q^{[k]*\beta} + t_k^{**} Q^{[k-1]**\beta}) + q_{(k)}^\alpha = 0$$
$$b_{\alpha\beta}^{(k)} N^{(k)\alpha\beta} - \nabla_\alpha^{(k)} (F^{(k)\alpha\beta} \varphi_\beta^{(k)}) + b_{\alpha\beta}^{(k)} F^{(k)\alpha\gamma} \psi_\gamma^{(k)\beta} -$$
$$\nabla_\alpha^{(k)} S^{(k)\alpha} + \nabla_\alpha^{(k)} (t_k^* Q^{[k]*\alpha} + t_k^{**} Q^{[k-1]**\alpha}) - \frac{1}{s}(t_k^* N^{[k]} - t_k^{**} N^{[k-1]}) - q_{(k)}^3 = 0$$
$$\nabla_\beta^{(k)} M^{(k)\alpha\beta} + S^{(k)\alpha} + \frac{h}{2s}(t_k^* Q^{[k]*\alpha} - t_k^{**} Q^{[k-1]**\alpha}) = 0$$

To obtain the linearized modifications of (3.5) and (3.10) it is sufficient to replace $\sigma^{\alpha\beta}$ by $F^{\alpha\beta}$ in the nonlinear terms and to consider the $\sigma^{\alpha\beta}$ left in the equations to be evaluated by using the linear expressions (3.1).

5. Application of the linearized equations to stability problems. As an illustration, let us consider the problem of the stability of a cylindrical sandwich shell subjected to longitudinal stress resultants. Let us limit ourselves to the case when the Kirchhoff-Love hypotheses are valid for stiff layers of radius R_α to which the stress resultants N are applied.

The equations (4.6) linearized relative to the membrane state, written in physical variables become for this case

$$\frac{\partial^2 u_\alpha}{\partial x^2} + \frac{1-\nu}{2} r_\alpha^2 \frac{\partial^2 u_\alpha}{\partial \varphi^2} + \frac{1+\nu}{2} r_\alpha \frac{\partial^2 v_\alpha}{\partial x \partial \varphi} +$$

$$\nu r_\alpha \frac{\partial w_\alpha}{\partial x} + \chi [(u_{\alpha+1} - u_\alpha)(1 + r_\alpha)\eta_{\alpha n} - (u_\alpha - u_{\alpha-1})(1 - r_\alpha)\eta_{\alpha 1}] +$$

$$\chi \left[\left(\frac{\partial w_\alpha}{\partial x} + \frac{\partial w_{\alpha+1}}{\partial x}\right)(1 + r_\alpha)\eta_{\alpha n} - \left(\frac{\partial w_{\alpha-1}}{\partial x} + \frac{\partial w_\alpha}{\partial x}\right)(1 - r_\alpha)\eta_{\alpha 1}\right] = 0$$

$$r_\alpha^2 \frac{\partial^2 v_\alpha}{\partial \varphi^2} + \frac{1-\nu}{2} \frac{\partial^2 v_\alpha}{\partial x^2} + \frac{1+\nu}{2} r_\alpha \frac{\partial^2 u_\alpha}{\partial x \partial \varphi} + r_\alpha^2 \frac{\partial w_\alpha}{\partial \varphi} + \chi [(v_{\alpha+1} - v_\alpha)(1 + r_\alpha)\times$$

$$\eta_{n\alpha} - (v_\alpha - v_{\alpha-1})(1 - r_\alpha)\eta_{\alpha 1}] + \chi \left[\left(r_\alpha \frac{\partial w_\alpha}{\partial \varphi} + r_{\alpha+1} \frac{\partial w_{\alpha+1}}{\partial \varphi}\right)(1 + r_\alpha)\eta_{\alpha n} -\right.$$

$$\left.\left(r_{\alpha-1} \frac{\partial w_{\alpha-1}}{\partial \varphi} + r_\alpha \frac{\partial w_\alpha}{\partial \varphi}\right)(1 - r_\alpha)\eta_{\alpha 1}\right] = 0 \quad (5.1)$$

$$
\begin{aligned}
d\left(\frac{\partial^4 w_\alpha}{\partial x^4}+2r_\alpha^2\frac{\partial^4 w_\alpha}{\partial x^2\partial\varphi^2}+r_\alpha^4\frac{\partial^4 w_\alpha}{\partial\varphi^4}\right)+r_\alpha^2 w_\alpha+r_\alpha^2\frac{\partial v_\alpha}{\partial\varphi}+\nu r_\alpha\frac{\partial u_\alpha}{\partial x}- \\
e\left[(w_{\alpha+1}-w_\alpha)(1+r_\alpha)\eta_{\alpha n}-(w_\alpha-w_{\alpha-1})(1-r_\alpha)\eta_{\alpha 1}\right]- \\
\chi\left[\left(\frac{\partial u_{\alpha+1}}{\partial x}-\frac{\partial u_\alpha}{\partial x}+\frac{\partial^2 w_\alpha}{\partial x^2}+\frac{\partial^2 w_{\alpha+1}}{\partial x^2}\right)(1+r_\alpha)\eta_{\alpha n}+\right. \\
\left.\left(\frac{\partial u_\alpha}{\partial x}-\frac{\partial u_{\alpha-1}}{\partial x}+\frac{\partial^2 w_{\alpha-1}}{\partial x^2}+\frac{\partial^2 w_\alpha}{\partial x^2}\right)(1-r_\alpha)\eta_{\alpha 1}\right]- \\
\chi r_\alpha\left[\left(\frac{\partial v_{\alpha+1}}{\partial\varphi}-\frac{\partial v_\alpha}{\partial\varphi}+r_\alpha\frac{\partial^2 w_\alpha}{\partial\varphi^2}+r_{\alpha+1}\frac{\partial^2 w_{\alpha+1}}{\partial\varphi^2}\right)(1+r_\alpha)\eta_{\alpha n}+\right. \\
\left.\left(\frac{\partial v_\alpha}{\partial\varphi}-\frac{\partial v_{\alpha-1}}{\partial\varphi}+r_{\alpha-1}\frac{\partial^2 w_{\alpha-1}}{\partial\varphi^2}+r_\alpha\frac{\partial^2 w_\alpha}{\partial\varphi^2}\right)(1-r_\alpha)\eta_{\alpha 1}\right]+N\frac{\partial^2 w_\alpha}{\partial x^2}=0
\end{aligned}
$$

$$(\alpha=1,\,2,\,3,\,\ldots,\,n)$$

The following dimensionless quantities are introduced here (the corresponding dimensional quantities are marked with an asterisk):

$$u_\alpha=v_1^{(\alpha)*}/c,\quad v_\alpha=v_2^{(\alpha)*}/c,\quad w_\alpha=w^{(\alpha)*}/c,\quad x=x^{1*}/c,\quad \varphi=x^{(\alpha)2*}/R_\alpha$$

$$r_\alpha=\frac{c}{R_\alpha},\quad d=\frac{1}{12}\frac{h^2}{c^2},\quad e=\frac{E_m c^2(1-\nu^2)}{Ehs},\quad \chi=\frac{G_m c^2(1-\nu^2)}{Ehs}$$

$$N=\frac{N^*(1-\nu^2)}{Eh}$$

The introduction of parametric terms in (5.1) is consistent with the accuracy of the hypotheses used, which correspond to the Donnell-Mushtari-Vlasov equations for a single-layered shell.

Let us consider the local buckling modes. We seek the solution of (5.1) in the form

$$w_\alpha=W_\alpha\sin kx\cos m\varphi,\quad v_\alpha=V_\alpha\sin kx\sin m\varphi,\quad u_\alpha=U_\alpha\cos kx\cos m\varphi \tag{5.2}$$

Substitution of (5.2) into (5.1) and introducing the n-dimensional vectors $\mathbf{u}_j$ ($j=1$, 2, 3) with the components U_α, V_α and W_α ($\alpha=1,2,3,\ldots,n$) results in a homogeneous system of equations

$$\mathbf{A}_{ij}\mathbf{u}_j-N\mathbf{B}_{i3}\mathbf{u}_3\delta_{i3}=0 \tag{5.3}$$

Because they are complicated, the expressions for the matrices $\mathbf{A}_{ij}$ and $\mathbf{B}_{33}$ are not presented here. Let us just note that the matrices $\mathbf{A}_{ij}$ are tridiagonal, and the matrix $\mathbf{B}_{33}$ is a diagonal matrix. Eliminating $\mathbf{u}_1$ and $\mathbf{u}_2$ from (5.3), we obtain

$$
\begin{gathered}
(\mathbf{A}-N\mathbf{B}_{33})\mathbf{u}_3=0,\qquad \mathbf{A}=\mathbf{A}_{33}+\mathbf{A}_{31}\mathbf{A}_u+\mathbf{A}_{32}\mathbf{A}_v \\
\mathbf{A}_u=(\mathbf{A}_{11}-\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{21})^{-1}(\mathbf{A}_{12}\mathbf{A}_{22}^{-1}\mathbf{A}_{23}-\mathbf{A}_{13}) \\
\mathbf{A}_v=(\mathbf{A}_{22}-\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12})^{-1}(\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{13}-\mathbf{A}_{23})
\end{gathered}
$$

Therefore, the bifurcation values of the stress resultants corresponding to the wave numbers k and m agree with the eigenvalues of the matrix $\mathbf{B}_{33}^{-1}\mathbf{A}$. Different eigenvalues for the same k and m correspond to different buckling modes in the transverse direction. Minimization of the values found in the wave numbers k and m yields a critical value of the stress resultant and the corresponding buckling mode.

Values of the bifurcation loads for cylindrical shells with a different number of layers, a different ratio of the moduli $f=E/E_m$ were found on the BESM-4 electronic computer. The remaining parameters were assumed to be the following: $h / c = 1$, $r_1 = 10$, $\nu = 0.3$, $E_m / G_m = 2.7$. Presented in Fig. 1 are results for shells having four stiff layers ($n = 4$). The solid lines show the least values of N_{km} for different m, while the dashes show higher values for $m = 0$. It turns out for this case that the critical dimensionless load $N_{kp} = 0{,}0599$, and that the corresponding buckling mode is axisymmetric: $m_* = 0$. The buckling mode in the longitudinal direction is characterized by the dimensionless wave number $k_* = 0.5$. The displacements are distributed over the thickness in such a manner that they diminish with distance from the outer surface of the shell, i.e. primarily surface buckling occurs. For shells with a large number of layers and the same parameters, the surface nature of the buckling becomes more explicit. The value of the critical load for a single-layered shell with the dimensions of the external layer are superposed by the dash-dot line in Fig. 1.

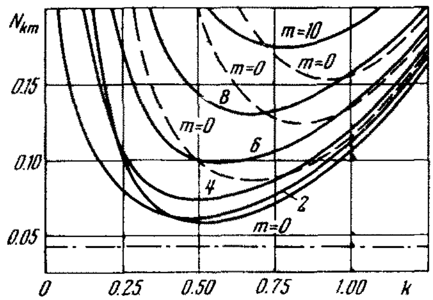

Fig. 1

The character of the buckling mode depends essentially on the stiffness of the soft layers. If the stiffness is quite low, then the buckling mode is similar to the buckling mode of a single-layered shell under axial compression. In practice, only the outer layer hence buckles. As the stiffness of the soft layers increases, the character of the buckling changes and in some range of variation of the stiffness the axisymmetric buckling mode corresponds to the minimum load. The outer layer hence behaves almost as a shell on an elastic foundation of Winkler type. This is seen from a comparison with the analysis of the expression for the bifurcation values of the load for a single-layered shell on a Winkler foundation

$$N_{km} = \frac{d\,(k^2 + r^2m^2)^2}{k^2} + (1 - \nu^2)\, r^2 \frac{k^2}{(k^2 + r^2m^2)^2} + \frac{4e}{k^2}$$

A further increase in the stiffness of the soft layers results in the axisymmetric buckling mode corresponds to the critical load.

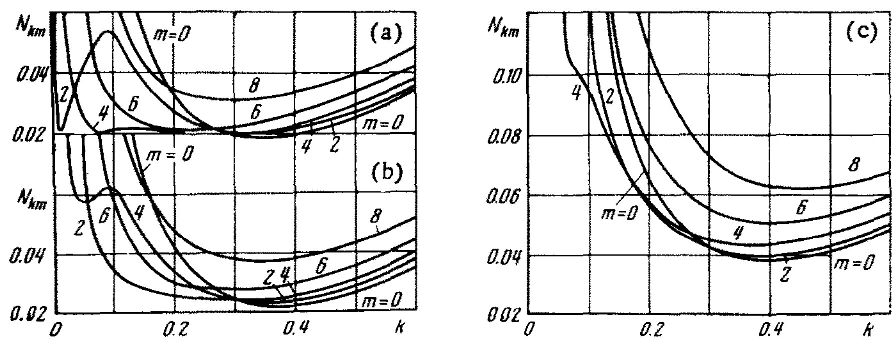

Fig. 2

The change in the buckling mode is shown in Fig. 2 a and b in the example of a shell with ten stiff layers ($n = 10$). Figure 2 a has been constructed for $f = 10^5$, Fig. 2 b for

$f = 10^3$, and Fig. 2 c for $f = 10^2$. The curves represented in Fig. 2 a are characteristic for a single-layered shell, and in Fig. 2 b and c for shells on an elastic basis. As the stiffness of the soft layers increases, the surface character of the buckling is spoiled more and more. The mode corresponding to the critical load is hence axisymmetric.

A further increase in the stiffness of the soft layers results in a new change in the character of the buckling. For a comparable stiffness of the "stiff" and "soft" layers, the shell starts to behave as a monolith, and the dependences of the bifurcation values of the loads have a form analogous to that presented in Fig. 2 a ; the buckling mode is again not axisymmetric.

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SMALL OSCILLATIONS OF A HEAVY RIGID BODY AROUND A FIXED POINT AND CERTAIN CASES OF EXISTENCE OF "LINEAR INTEGRALS"

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F. Kh. TSEL'MAN
(Moscow)
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A large amount of literature (for example, see the surveys in [1, 2]) has been devoted to the motion of a heavy rigid body around a fixed point. The present paper is based on a simple concept, permitting us to use the methods of investigating systems with nonlinearly connected oscillators [3–5] for the study of a specific Hamiltonian system with three degrees of freedom, namely, a rigid body moving around a fixed point. This concept is that when no constraints are imposed on the initial conditions, excluding small motions near the equilibrium position (and such motions are all the general cases of integrability: Euler-Poinsot,